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# A Comprehensive Introduction of the Finite Element Method for Undergraduate Courses

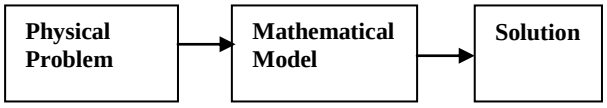
Pedro Americo Almeida Magalhaes Junior; Daniel Januário Cordeiro Gomes.

***Abstract***
A simple and comprehensive introduction of the Finite Element Method for undergraduate courses is proposed. With very simple mathematics, students can easily understand it. The primary objective is to make students comfortable with the approach and cognizant of its potentials. The technique is based on the general overview of the steps involved in the solution of a typical finite element problem. This is followed by simple examples of mathematical developments which allow developing and demonstrating the major aspects of the finite element approach unencumbered by complicating factors.

***Index Terms***– Finite Element Method, Undergraduate Courses, Numerical Analysis, Teaching Methods and Technique.

## I. INTRODUCTION

In a simplified view, the engineer transforms a physical problem into a mathematical model using the knowledge of the laws of the physics and chemistry. It is necessary that this model pass a resolution phase to reach a solution. In this phase, the numerical methods enter predominantly where great limitations in the analytical and algebraic mathematical methods exist. Nowadays, in addition to the high complexity of practical problems, there is an increasing demand for effectiveness from modern engineering designs. In this circumstance, numerical solutions are the only alternatives available.




Modeling Phase Resolution Phase
Fig. 1 - Phases for the resolution of a problem.

Designers of electrical equipment for power applications have to satisfy the customer in a number of points and aim, of course, to do so better than the competition. Typical criteria, a number of which will appear in combination in most cases, include: low operating cost, low initial cost, high efficiency, high reliability, minimum weight, volume or diameter, close tolerances to performance specifications and parameters, tolerance to occasional severe abnormal conditions, ability to generate voltage with low harmonic distortion and ability to operate satisfactorily from a non-sinusoidal supply.

New types of machines are being developed and applied, such as permanent magnet machines and reluctance motors. In defense applications, leakage fields and magnetic signatures may be important. Leakage fields may also be important in traction applications which humans may be close to the drive equipment. Electromagnetic interference is of growing importance.

It is clear that in many, if not most cases, it is essential to be able to analyze any proposed design in considerable detail, so that near optimal results may be obtained. This becomes especially important for large or special-purpose equipment with which cut-and-try methods are impossible or prohibitively expensive. Many of the critical factors mentioned above are dependent on magnetic and electric field distribution and the calculation of these quantities with the accuracy now required cannot be carried out by analytical procedures. The difficulties born by factors such as complicated geometry, saturation effects in iron, the presence of solid material in which eddy currents can be induced, and in some cases, three-dimensional effects, mean that a numerical method is required. [8]

If a designer is to achieve the aims outlined above, it is needed a design tool which will take the following into account: irregular geometric shapes of components, non-linearity of magnetic or electric materials, induced currents in non-regular components, anisotropic materials or structures, external circuits, integration of thermal and mechanical effects and non-sinusoidal time variation of currents and fields.

Obviously the list above eliminates analytic techniques unable to deal with irregular geometry and non-linear materials. However, numerical methods are capable of modeling the true geometry of the design in two and three dimensions. Non-linear characteristics of materials can be allowed for through iterative schemes such as the Newton-Raphson method and the non-sinusoidal time variation represented by time-stepping techniques. Anisotropic materials and structures, induced

currents and external circuits can also be included in numerical treatments.

## II. FINITE ELEMENT METHOD

Having established that numerical techniques are essential for an ease of advanced design, the question is which one to choose. Several methods have been known for some 40 years and the advent of fast digital computers has encouraged and facilitated their development. The main numerical methods available are: finite difference, finite element and boundary element.

All have their advantages and disadvantages, a fact which makes a choice difficult. However, the Finite Element Method (FEM) incorporates most of the advantages of the two other techniques without incurring into significant disadvantages. For instance, the finite difference technique is not easily applicable to modeling irregular geometries which include slants and curved surfaces. Nodal distribution can be very inefficient. This is not so with finite elements. Equally, the boundary element method can efficiently model regions in which the material properties are linear, but it is not efficient at handling non-linear materials. Again, FEM is well suited to modeling non-linear materials. Thus, without considering the pros and cons of each method in detail, it is accepted that the FEM is important to the needs of designers.

In contrast to finite-difference techniques, the FEM divides the solution domain into simply shaped regions, or elements. An approximate solution for partial differential equations can be developed for each of these elements. Linking together, or assembling these individual solutions ensures the continuity at the interelement boundaries to generate the total solution. Thus, the partial differential equations are satisfied in a piecewise fashion. The use of elements, rather than a rectangular grid, provides a much better approximation for irregularly shared systems. Further, unknown values can be generated continuously across the entire solution domain rather than at isolated points.

Although the particulars will vary, the implementation of the finite-element approach usually follows a standard step-by-step procedure. The following provides a brief overview of each of these steps. The steps for the application of the finite elements methods are:

**Preprocessing:**

- *Definition of the problem and the domain*
- *Discretization or division of the domain into elements*
  - One-Dimension (Line element)
  - Two Dimensions (Triangular or Quadrilateral element)
  - Three Dimensions (Tetrahedron or Hexahedron element)

**Processing:**

- *To get the Element Equations [k]{u}={f}*
  - Choice of Approximation Functions (linear functions-straight and plans)
  - Obtaining an Optimal Fit of Approximation Functions to the Solution
    - Direct Approach
    - Method of Weighted Residuals
      - Collocation Approach
      - Sub domain Method
      - Least-Squares Methods
      - Galekin's Methods
    - Variational techniques
      - Rayleigh-Ritz's Method
- *Assembly or Link Together the Element Equations [K]{U}={F}*
- *Incorporation of the Initial and Boundary Conditions [K']{U'}={F'}*
- *Solution of the linear system (or not linear). {U'}*

**Post processing**:

- *Presentation of results or graphical visualization*
- *Determination of secondary variable.*

Where: - Lowercase terms:
*[k]* = an element property or stiffness matrix,
*{u}*= a column vector unknowns at the nodes and,
*{f}*= a column vector reflecting the effect of any external influences applied at the nodes.
- Uppercase terms:
*[K]* = the assemblage property matrix,
*{U}* = Column vectors for unknowns,
*{F}* = Column vectors for external forces and
*[K'], {U'}* and *{F'}* = boundary conditions have been incorporated.

## III. CHOICE OF APPROXIMATION FUNCTIONS. [1]

Because it is easy to manipulate mathematically, polynomials are often employed for this purpose. For the one-dimensional case, the simplest alternative is a first order polynomial or straight line,

$$u(x) = a_0 + a_1 x \tag{1}$$

Where u (x) = the dependent variable, $a_0$ and $a_1$ = constants, and x = the independent variable. This function must pass through the values of u (x) at the end points of the element at $x_1$ and $x_2$. Therefore,

$$u_1 = a_0 + a_1 x_1 \qquad u_2 = a_0 + a_1 x_2$$

Where $u_1 = u(x_1)$ and $u_2 = u(x_2)$. These equations can be solved using Cramer's rule for

$$a_0 = (u_1 x_2 - u_2 x_1)/(x_2 - x_1) \quad a_1 = (u_2 - u_1)/(x_2 - x_1)$$

These results can then be substituted into Eq. (1), which, after collection of terms, can be written as

$$u = N_1 u_1 + N_2 u_2 \tag{2}$$

Where

$$N_1 = (x_2 - x)/(x_2 - x_1)$$

(3)

and

$$N_2 = (x - x_1)/(x_2 - x_1) \quad (4)$$

Equation (2) is called an approximation, or shape, function and $N_1$ and $N_2$ are called interpolation functions. Close inspection reveals that Eq. (2) is, in fact, the Lagrange first-order interpolating polynomial. It provides a means to predict intermediate values (that is, to interpolate) between given values $u_1$ and $u_2$ at the nodes.

In addition, the fact that it is dealing with linear equation facilitates operations such as differentiation and integration. The derivative of Eq. (2) is

$$\frac{du}{dx} = \frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2 \quad (5)$$

According to Eqs. (3) and (4), the derivatives of the N's can be calculated as

$$\frac{dN_1}{dx} = -\frac{1}{x_2 - x_1} \qquad \frac{dN_2}{dx} = \frac{1}{x_2 - x_1} \quad (6)$$

and, therefore, the derivative of u is

$$\frac{du}{dx} = \frac{1}{x_2 - x_1}(-u_1 + u_2) \quad (7)$$

In other words, it is a divided difference representing the slope of the straight line connecting the nodes. The integral can be expressed as

$$\int_{x_1}^{x_2} u dx = \int_{x_1}^{x_2} N_1 u_1 + N_2 u_2 \; dx$$

Each term on the right-hand side is merely the integral of right triangle with base $x_2$-$x_1$ and height u. That is,

$$\int_{x_1}^{x_2} N u dx = \frac{1}{2}(x_2 - x_1)u$$

Thus, the entire integral is

$$\int_{x_1}^{x_2} u dx = \frac{u_1 + u_2}{2}(x_2 - x_1) \quad (8)$$

In other words, it is simply the trapezoidal rule.

Just as for the one-dimensional case, the next step is to develop an equation to approximate the solution for the element. For a triangular element, the simplest approach is the linear polynomial.

$u(x, y) = a_0 + a_{1,1}x + a_{1,2}y$

where u(x, y) = the dependent variable, a's = coefficients, and x and y = independent variables. This function must pass through the values of u(x,y) at the triangle's nodes ($x_1$, $y_1$), ($x_2$, $y_2$), and ($x_3$, $y_3$). Therefore,

$u_1(x, y) = a_0 + a_{1,1}x_1 + a_{1,2}\, y_1$

$u_2(x, y) = a_0 + a_{1,1}x_2 + a_{1,2}\, y_2$

$u_3(x, y) = a_0 + a_{1,1}x_3 + a_{1,2}\, y_3$

or in matrix form,

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_{1,1} \\ a_{1,2} \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

which can be solved for

$$a_0 = \frac{1}{2A_e}[u_1(x_2 y_3 - x_3 y_2) + u_2(x_3 y_1 - x_1 y_3) + u_3(x_1 y_2 - x_2 y_1)]$$

$$a_{1,1} = \frac{1}{2A_e}[u_1(y_2 - y_3) + u_2(y_3 - y_1) + u_3(y_1 - y_2)]$$

$$a_{1,2} = \frac{1}{2A_e}[u_1(x_3 - x_2) + u_2(x_1 - x_3) + u_3(x_2 - x_1)]$$

where $A_e$ is the area of the triangular element:

$$A_e = \frac{1}{2}[(x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1)]$$

Equations above can be substituted into u(x,y). After collection of terms, the result can be expressed as

$u = N_1u_1 + N_2u_2 + N_3u_3$

where

$$N_1 = \frac{1}{2A_e}[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$N_2 = \frac{1}{2A_e}[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

$$N_3 = \frac{1}{2A_e}[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

Equation $u = N_1u_1 + N_2u_2 + N_3u_3$ provides a means to predict intermediate values for the element on the basis of the values at its nodes. Figure 2 shows the shape function along with the corresponding interpolation functions. Notice that the sum of the interpolation functions is always equal to 1. As with the one-dimensional case, various methods are available for developing element equations based on the underlying PDE and the approximating functions. The resulting equations are considerably more complicated than the one-dimensional case. However, because the approximating functions are usually lower-order polynomials, the terms of the final element matrix will consist of lower-order polynomials and constants.

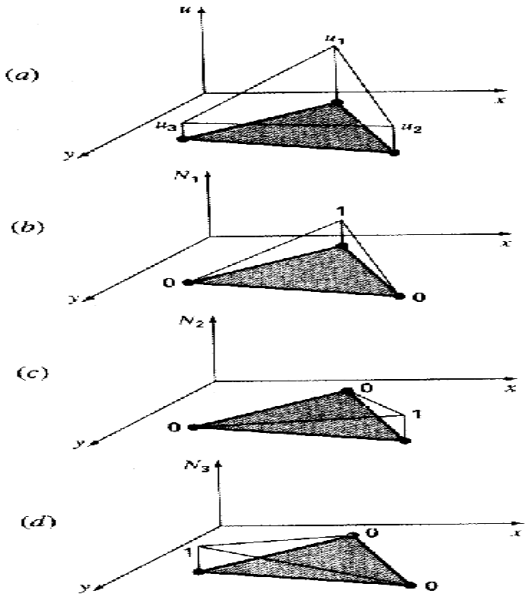



Fig. 2 : A linear approximation function for a triangular element. The corresponding interpolation functions are show in (b) through (d).[1]

## IV. THE METHODS OF WEIGHTED RESIDUALS.

The differential equations, for example

$$\frac{d^2T}{dx^2} = -f(x) \tag{9}$$

can be expressed as :

$$\frac{d^2T}{dx^2} + f(x) = 0 \tag{10}$$

The approximate solution (straight line) can be substituted into this equation. Because the approximation function is not the exact solution, the left side of the resulting equation will not be zero but will equal a residual,

$$R = \frac{d^2T}{dx^2} + f(x) \tag{11}$$

The methods of weighted residuals consists of finding a minimum for the residual according to the general formula

$$\int_D RW_i dD = 0 \qquad i = 1, 2, \ldots, m \tag{12}$$

where D = the solution domain and the $W_i$ = linearly independent weighting functions.

At this point, there are a variety of choices that could be made for the weighting function. Several choices can be made for the weighting functions of Eq. (12). Each represents an alternative approach for the method.

In the collocation approach, it chooses as many locations as there are unknown coefficients. Then, the coefficients are adjusted until the residual vanishes at each of these locations. Consequently, the approximating function will yield prefect results at the chosen locations but will have a nonzero residual elsewhere. Thus, it is akin to the interpolation methods. Note that collocation amounts to using the weighting function

$$W_i = \delta(x - x_i) \qquad para\ i = 1,2,..., n \tag{13}$$

where n = the number of unknown coefficients and $\delta(x-x_i)$ = the Dirac delta function that vanishes everywhere but at $x=x_i$, where it equals 1.

In the sub domain methods the interval is divided into as many segments, or sub domains, as there are unknown coefficients. Then, the coefficients are adjusted until the average value of the residual is zero in each sub domain. Thus, for each sub domain, the weighting function is equal to 1 and Eq. (12) is

$$\int_{x_{i-1}}^{x_i} R dx = 0 \qquad para\ i = 1,2,\ldots,n \tag{14}$$

Where $x_{i-1}$ and $x_i$ are the bounds of the sub domain.

For the least-square case, the coefficients are adjusted so as to minimize the integral of the square of the residual. Thus, the weighting functions are

$$W_i = \frac{\partial R}{\partial a_i} \tag{15}$$

which can be substituted into Eq. (12) to give

$$\int_D R\frac{\partial R}{\partial a_i} dD = \frac{\partial}{\partial a_i}\int_D R^2 dD = 0 \qquad i = 1,2,..., n \tag{16}$$

Comparison of the formulation shows that this is the continuous form of regression.

Galerkin's methods employ the interpolation functions $N_i$ as weighting function. Recall that these functions always sum to 1 as any position in an element. For many problem contexts. Galerkin's methods yield the same results as are obtained by variational methods. Consequently, it is the most commonly employed version of methods weighting residual used in finite-element analysis.

## V. THE ELLIPTIC EQUATION [2]

The basic elliptic equation is

$$-\nabla.(c\nabla u) + au = f \quad \text{in} \quad \Omega \tag{17}$$

where Ω is a bounded domain in the plane (xy). *c*, *a*, *f*, and the unknown solution *u* are complex functions defined on Ω. *c* can also be a N-by-N matrix function on Ω [2]. The boundary conditions specify a combination of *u* and its normal derivative on the boundary:

*Dirichlet: hu = r* on the boundary ∂Ω.

*Generalized Neumann:* $\vec{n}.(c\nabla u) + qu = g$ on ∂Ω.

*Mixed:* Only applicable to *systems*. A combination of Dirichlet and generalized Neumann.

$\vec{n}$ is the outward unit normal. *g*, *q*, *h*, and *r* are functions defined on ∂Ω.

Its nomenclature deviates slightly from the tradition for potential theory, where a Neumann condition usually refers to the case $q = 0$ and its Neumann would be called a mixed condition. In some contexts, the generalized Neumann boundary conditions are also referred to as the *Robin boundary condition*s. In variational calculus, Dirichlet conditions are also called essential boundary conditions and restrict the trial space. Neumann conditions are also called natural conditions and arise as necessary conditions for a solution. The variational form of the equation with Neumann conditions is given below.

The approximate solution to the elliptic PDE (Partial Differential Equations) is found in three steps:

a) Describe the geometry of the domain Ω and the boundary conditions.
b) Build a triangular mesh on the domain Ω. It has mesh generating and mesh refining facilities. A mesh is described by three matrices of fixed format that contain information about the mesh points, the boundary segments, and the triangles.
c) Discretize the PDE and the boundary conditions to obtain a linear system $Ku = F$. The unknown vector $u$ contains the values of the approximate solution at the mesh points, the matrix $K$ is assembled from the coefficients $c$, $a$, $h$, and $q$ and the right-hand side $F$ contains, essentially, averages of $f$ around each mesh point and contributions from $g$. Once the matrices $K$ and $F$ are assembled, it can to solve the linear system and further process the solution.

More elaborate applications make use of the FEM-specific information returned by the different functions . Therefore it quickly summarizes the theory and technique of FEM solvers to enable advanced applications to make full use of the computed quantities. FEM can be summarized in the following sentence: *Project the weak form of the differential equation onto a finite-dimensional function spac*e. The rest of this section deals with explaining the above statement.

It starts with the *weak form of the differential equatio*n. Without restricting the generality, it assumes generalized Neumann conditions on the whole boundary, since Dirichlet conditions can be approximated by generalized Neumann conditions. In the simple case of a unit matrix $h$, setting $g = qr$ and then letting $q \to \infty$ yields the Dirichlet condition because division with a very large $q$ cancels the normal derivative terms.

Assume that $u$ is a solution of the differential equation. Multiply the equation with an arbitrary *test function v* and integrate on Ω:

$$\int_{\Omega} -(\nabla.(c\nabla u))v + auv \; d\Omega = \int_{\Omega} fv \; d\Omega \qquad (18)$$

Integrate by parts (i.e., use Green's formula) and the boundary integral can be replaced by the boundary condition:

$$\int_{\Omega} (c\nabla u).\nabla v + auv \, d\Omega - \int_{\partial\Omega} (-qu + g)v \, d\partial\Omega = \int_{\Omega} fv \, d\Omega \qquad (19)$$

Replace the original problem with: *Find u such that*

$$\int_{\Omega} (c\nabla u).\nabla v + auv - fv d\Omega - \int_{\partial\Omega} (-qu + g)v d\partial\Omega = 0, \forall v \qquad (20)$$

This equation is called the variational, or weak, form of the differential equation. Obviously, any solution of the differential equation is also a solution of the variational problem. The reverse is true under some restrictions on the domain and on the coefficient functions. The solution of the variational problem is also called the weak solution of the differential equation.

The solution $u$ and the test functions $v$ belong to some function space V. The next step is to choose an $N_p$ -dimensional subspace $V_{N_p} \subset V$ . *Project the weak form of the differential equation onto a finite-dimensional function space* simply means requesting $u$ and $v$ to lie in $V_{N_p}$ rather than V. The solution of the finite dimensional problem turns out to be the element of $V_{N_p}$ that lies closest to the weak solution when measured in the energy norm (see below). Convergence is guaranteed if the space $V_{N_p}$ tends to $V$ as $N_p \to \infty$ . Since the differential operator is linear, it demands that the variational equation is satisfied for $N_p$ test-functions $\phi_i \in V_{N_p}$ that form a basis, i.e.,

$$\int_{\Omega} (c\nabla u).\nabla \phi_i + au\phi_i - f\phi_i \; d\Omega - \int_{\partial\Omega} (-qu + g)\phi_i \; d\partial\Omega = 0, \; \mathrm{i} = 1,\ldots,\mathrm{N_p} \qquad (21)$$

Expand $u$ in the same basis of $V_{N_p}$

$$u(x, y) = \sum_{j=1}^{N_p} U_j \phi_j(x, y)$$

and obtain the system of equations

$$\sum_{j=1}^{N_p} \left( \int_{\Omega} (c\nabla \phi_j).\nabla \phi_i + a\phi_j\phi_i d\Omega + \int_{\partial\Omega} q\phi_j\phi_i \; d\partial\Omega \right) U_j = \int_{\Omega} f\phi_i d\Omega + \int_{\partial\Omega} g\phi_i d\partial\Omega, \quad \mathrm{i} = 1,\ldots,\mathrm{N_p} \qquad (22)$$

Use the following notations:

$$K_{i,j} = \int_{\Omega} (c\nabla\phi_j).\nabla\phi_i d\Omega \quad \text{(Stiffness matrix)}$$

$$\mathrm{M}_{\mathrm{i,j}} = \int_{\Omega} a\phi_j\phi_i d\Omega \qquad \text{(Mass matrix)}$$

$$\mathrm{Q}_{\mathrm{i,j}} = \int_{\partial\Omega} q\phi_j\phi_i d\partial\Omega \quad F_i = \int_{\Omega} f\phi_i d\Omega \quad G_i = \int_{\partial\Omega} g\phi_i d\partial\Omega$$

and rewrite the system in the form $(K + M + Q)U = F + G$. K, M, and $Q$ are $N_p$ -by-$N_p$ matrices, and $F$ and $G$ are $N_p$ -vectors. When it is not necessary to distinguish K, M, and $Q$ or $F$ and G, it collapses the notations to $KU = F$. [2]

When the problem is *self-adjoint* and *elliptic* in the usual mathematical sense, the matrix $K + M + Q$ becomes symmetric and positive definite. Many common problems have these characteristics, most notably those that can also be formulated as minimization problems. For the case of a scalar equation, K, M, and $Q$ are obviously symmetric. If $c(x,y) \geq \delta > 0$, $a(x,y) \geq 0$ and $q(x,y) \geq 0$ with $q(x,y) > 0$ on some part of $\partial\Omega$, then, if $U \neq 0$.

$$U^T(K + M + Q)U = \int_{\Omega} c|\nabla u|^2 + au^2 d\Omega + \int_{\partial\Omega} qu^2 d\partial\Omega > 0, \text{ if } U \neq 0. \tag{23}$$

$U^T(K + M + Q)U$ is the *energy nor*m. There are many choices of the test-function spaces. It uses continuous functions that are linear on each triangle of the mesh. Piecewise linearity guarantees that the integrals defining the stiffness matrix $K$ exist. Projection onto $V_{N_p}$ is nothing more than linear interpolation, and the evaluation of the solution inside a triangle is done just in terms of the nodal values. If the mesh is uniformly refined, $V_{N_p}$ approximates the set of smooth functions on Ω.

A suitable basis for $V_{N_p}$ is the set of "tent" or "hat" functions $\phi_i$ . These are linear on each triangle and take the value 0 at all nodes $x_j,y_j$ except for $x_i,y_i$ . Requesting $\phi_i(x_j,y_j) = 1$ yields the very pleasant property

$$u(x_i, y_i) = \sum_{j=1}^{N_p} U_j\phi_j(x_i, y_i) = U_i, \tag{24}$$

i.e., by solving the FEM system it obtains the nodal values of the approximate solution. Finally note that the basis function $\phi_i$ vanishes on all the triangles that do not contain the node $x_i$ . The immediate consequence is that the integrals appearing in $K_{i,j}$ , $M_{i,j}$, $Q_{i,j}$, $F_i$ and $G_i$ only need to be computed on the triangles that contain the node $x_i$ . Secondly, it means that $K_{i,j}$ and $M_{i,j}$ are zero unless $x_i,y_i$ and $x_j,y_j$ are vertices of the same triangle and thus $K$ and $M$ are very sparse matrices. Its sparse structure depends on the ordering of the indices of the mesh points.

The integrals in the FEM matrices are computed by adding the contributions from each triangle to the corresponding entries (i.e., only if the corresponding mesh point is a vertex of the triangle). This process is commonly called *assembling*.

The assembling scans the triangles of the mesh. For each triangle it computes the so-called local matrices (The local 3-by-3 matrices contain the integrals evaluated only on the current triangle. The coefficients are assumed constant on the triangle and it is evaluated only in the triangle barycenter.) and add its components to the correct positions in the sparse matrices or vectors. The integrals are computed using the mid-point rule. This approximation is optimal since it has the same order of accuracy as the piecewise linear interpolation.

Consider a triangle given by the nodes $P_1$, $P_2$, and $P_3$ as in the following figure 3.

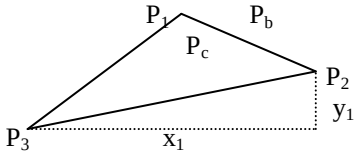



Fig. 3: The local triangle $P_1P_2P_3$ [2].

The simplest computations are for the local mass matrix m:

$$m_{i,j} = \int_{\Delta P_1P_2P_3} a(P_c)\phi_i(x,y)\phi_j(x,y)d\Delta = a(P_c)\frac{area(\Delta P_1P_2P_3)}{12}(1+\delta_{i,j}), \tag{25}$$

where $Pc$ is the center of mass of $\Delta P_1P_2P_3$ , i.e.,

$$P_c = \frac{P_1 + P_2 + P_3}{3}$$

The contribution to the right-hand side $F$ is just

$$f_i = f(P_c)\frac{area(\Delta P_1P_2P_3)}{3} \tag{26}$$

For the local stiffness matrix it has to evaluate the gradients of the basis functions that do not vanish on $P_1P_2P_3$. Since the basis functions are linear on the triangle $P_1P_2P_3$, the gradients are constants. Denote the basis functions $\phi_1$, $\phi_2$, and $\phi_3$ such that $\phi_i(Pi) = 1$. If $P_2 - P_3 = [x_1,y_1]^T$ then it has that

$$\nabla\phi_1 = \frac{1}{2area(\Delta P_1P_2P_3)}\begin{bmatrix} y_1 \\ -x_1 \end{bmatrix} \tag{27}$$

and after integration (taking $c$ as a constant matrix on the triangle) [2]

$$k_{i,j} = \frac{1}{4area(\Delta P_1P_2P_3)}\begin{bmatrix} y_j, -x_j \end{bmatrix} c(P_c)\begin{bmatrix} y_1 \\ -x_1 \end{bmatrix} \tag{28}$$

If two vertices of the triangle lie on the boundary $\partial\Omega$, it contributes to the line integrals associated to the boundary conditions. If the two boundary points are $P_1$ and $P_2$ , then it has

$$Q_{i,j} = q(P_b)\frac{\|P_1 - P_2\|}{6}(1+\delta_{i,j}), \quad i,j = 1,2 \qquad (29)$$

and

$$G_i = g(P_b)\frac{\|P_1 - P_2\|}{2}, \quad i,j = 1,2 \qquad (30)$$

where $P_b$ id the mid-point of $P_1P_2$.

For each triangle the vertices $P_m$ of the local triangle correspond to the indices $i_m$ of the mesh points. The contributions of the individual triangle are added to the matrices such that, e.g.,

$$K^t_{i_m,i_n} \leftarrow K_{i_m,i_n} + k_{m,n}, \quad m,n = 1,2,3 \qquad (31)$$

The Dirichlet boundary conditions are treated in a slightly different manner. It is eliminated from the linear system by a procedure that yields a symmetric, reduced system. It can return matrices *K, F, B,* and *ud* such that the solution is *u* = *Bv* + *ud* where *Kv* = F. *u* is an $N_p$ -vector, and if the rank of the Dirichlet conditions is *r*D, then *v* has $N_p - rD$ components.

## VI. THE PARABOLIC EQUATION [2]

The elliptic solver allows other types of equations to be more or less easily implemented. Below it shows how the parabolic equation can be reduced to solving elliptic equations. This is done by the function parabolic [2]. Consider the equation

$$d\frac{\partial u}{\partial t} - \nabla.(c\nabla u) + au = f \quad \text{in} \quad \Omega \qquad (32)$$

with the initial condition

$$u(x,y,0) = u_0(x,y) \qquad x,y \in \Omega$$

and boundary conditions of the same kind as for the elliptic equation on ∂Ω.

The heat equation reads

$$\rho C\frac{\partial u}{\partial t} - \nabla.(k\nabla u) + h(u - u_\infty) = f \qquad (33)$$

in the presence of distributed heat loss to the surroundings. ρ is the density, *C* thermal capacity, *k* thermal conductivity, *h* film coefficient, $u_\infty$ ambient temperature, and *f* heat source.

For time-independent coefficients, the steady state solution of the equation is the solution to its standard elliptic equation

$$-\nabla.(c\nabla u) + au = f. \qquad (34)$$

Assuming a triangular mesh on Ω and at any time $t \geq 0$, expand the solution to the partial differential equations (as a function of x,y) in the Finite Element Method basis:

$$u(x,y,t) = \sum_i U_i(t)\phi_j(x,y) \qquad (35)$$

Plugging the expansion into the PDE, multiplying with a test function $\phi_j$, integrating over Ω and applying Green's formula and the boundary conditions yield:

$$\sum_i \int_\Omega d\phi_j\phi_i d\Omega \frac{dU_i(t)}{dt} +$$
$$\sum_i \left( \int_\Omega \nabla\phi_j.(c\nabla\phi_i) + a\phi_j\phi_i d\Omega + \int_{\partial\Omega} q\phi_j\phi_i d\partial\Omega \right) U_i(t)$$
$$= \int_\Omega f\phi_j d\Omega + \int_{\partial\Omega} g\phi_j d\partial\Omega \qquad \forall j. \qquad (36)$$

In matrix notation, it has to solve the *linear, large* and *sparse* ordinaries differential equations system

$$M\frac{dU}{dt} + KU = F. \qquad (37)$$

This method is traditionally called *method of lines* semidiscretization.

Solving the ODE (Ordinaries Differential Equations) with the initial value

$$U_i(0) = u_0(x_i, y_i) \qquad (38)$$

yields the solution to the PDE at each node $x_i$,$y_i$ and time *t.* Note that *K* and *F* are the stiffness matrix and the right-hand side of the elliptic problem

$$-\nabla.(c\nabla u) + au = f \quad \text{in} \quad \Omega \qquad (39)$$

With the original boundary conditions while *M* is just the mass matrix of the problem

$$-\nabla.(0\nabla u) + du = 0 \quad \text{in} \quad \Omega. \qquad (40)$$

When the Dirichlet conditions are time dependent, *F* contains contributions from time derivatives of *h* and r. These derivatives are evaluated by finite differences of the user-specified data. [2]

The ODE system is ill conditioned. Explicit time integrators are forced by stability requirements to very short time steps while implicit solvers can be expensive since it solves an elliptic problem at every time step. The numerical integration of the ODE system is performed. Suite functions, which are efficient for this class of problems. The time-step is controlled to satisfy a tolerance on the error, and factorizations of coefficient matrices are performed only when necessary. When coefficients are time dependent, the necessity of re-evaluating and re-factorizing the matrices each time-step may still make the solution time consuming, although parabolic re-evaluates only that which varies with time. In certain cases a time-dependent Dirichlet matrix h(t) may cause the error control to fail, even if the problem is mathematically sound and the solution u(t) is smooth. This can happen because the ODE integrator looks only at the reduced solution *v* with *u* = *Bv* + *u*d. As *h* changes, the pivoting scheme employed for numerical stability may change the elimination order from one step to the next. This means that *B*,*v* and *ud* all change discontinuously, although *u* itself does not. [2]

Using the same ideas as for the parabolic equation, hyperbolic implements the numerical solution of with

the initial conditions and usual boundary conditions. In particular, solutions of the equation $u_{tt} - c\nabla u = 0$ are waves moving with speed $\sqrt{c}$ .

## VII. CONCLUSION

With the fast scientific and technological advance of the last years, many subjects have become sufficiently complex and accessible only to a small number of specialists in the area. For the scientific development to continue, it is essential new study techniques and methodologies fit to passing this knowledge on in an easy and fast way to make it accessible to a great number of people, either professionals or students.

There are no readymade teaching formulas, each lesson is a unique and brief process of interaction between teachers and students, because it depends on their personal features. However, methodologies, suggestions and experiences can be gathered to help teaching, thus facilitating the process of knowledge transference.

Initially, FEM was conceived as a mathematical artifice for the calculation of structures, and later became a powerful instrument for the resolution of partial derivatives equations, so common in the physics of the continuous bodies. Either in civil, electrical, mechanical engineering or in problems of movement of solid or fluid bodies in thermodynamics, the electromagnetism of the static fields or of propagation, FEM represents an almost universal tool for the pre-determination of the physical behavior of the objects under study. The recent development of CAE techniques (Computer Aided Engineering) has allowed the integration of the different algorithms of calculation and their association with the modern interactive graphical methods to free the engineer from all the tiring tasks of programming. Inside this integration, FEM uses all its power to facilitate the direct passage from the model to the conception.

This article presents forms of teaching FEM in undergraduate engineering courses. It propose a new teaching technique to approach of the necessary mathematics and its use in an elementary way. It is suggested that the subject be given in courses such as Numerical Analysis, Numerical Calculus and Numerical Methods.

To sum up, the article presents two conclusions: first, that it is important to develop efficient teaching methods for FEM as early as in engineering undergraduate courses, and second, that FEM represents an important advance in the syllabus of disciplines like numerical analysis, numerical calculus and numerical methods in undergraduate courses.

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**Pedro Americo Almeida Magalhaes Junior** has a bachelor degree in Electrical Engineer from Universidade Federal de Minas Gerais (UFMG) and Computer Science by Pontificia Universidade Catolica de Minas Gerais (PUC-MG), Belo Horizonte, Brazil, 1989. He holds a specialist diploma in Material Engineering from Fundacao Educacional Minas Gerais (FEMG), 1995. He is currently a Numeric Methods teacher at the Mathematics Department of PUC-MG, Brazil. He is presently working on an Electrical Engineer MS at PUC-MG. His research interests include Numeric Methods, Finite Element Method, Electromagnetic and Teaching Methods. (Email: pamerico@bdmg.mg.gov.br)

**Daniel Januário Cordeiro Gomes** has a bachelor degree in Mechanical Engineer form Universidade Federal de Minas Gerais (UFMG), 2008. He has a master's degree in Mechanical Engineer by Pontificia Universidade Catolica de Minas Gerais (PUC-MG), 2011. He is currently professor with emphasis on Machine Components, Structural Analysis and Mechanics of Materials. He is presently working at the college Centro Universitário UNA, Belo Horizonte, Brazil. (Email: danieljanuario@yahoo.com.br).